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Use of the Wiener–Hermite expansion for nearly normal turbulence

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The turbulence problem is formulated using the Wiener stochastic expansion. The expansion is useful for processes which are in some sense nearly normal, and can be used for non-linear non-Gaussian processes such as many turbulent fluid flows. Here we present the general formulation for statistically inhomogeneous and anistropic processes.

The transfer term in the energy equation, or equivalently the third-order velocity correlation, forms a sensitive measure of the amount of non-Gaussianity present in real fluid flows. Experimental evidence shows that in many flows this component is small compared with the Gaussian part. It is shown that a homogeneous and isotropic flow which has but a small non-Gaussian part possesses a distribution at one point which is Gaussian to terms of second order. The experiments suggest that immediately behind a grid in a wind tunnel the flow is very nearly normal. The non-Gaussian part grows at a moderate rate, at least within the range of distance downstream (or decay time) available in the usual experiments. This growth is probably due to the relative increase in the amount of energy in the smallest eddies, which are non-normal.

A necessary criterion for the validity of the zero-fourth-cumulant approximation is suggested: the transfer term in dimensionless form should be small. It is shown that calculations using the zero-fourth-cumulant approximation have given negative energy spectra when this condition is violated, probably for the reason that the process is no longer nearly Gaussian. However, even when this condition is fulfilled, it is shown that that approximation must be corrected.

It is suggested that the present theory is valid for quite large times of decay if initial energy spectra are chosen which are not too far from the actual physical values for fluid in turbulent flow. Equations are given for the next-higher-order term in a nearly normal approximation. The expansion is also used in § 6 to describe turbulent mixing problems and is compared with the zero-fourth-cumulant approximation for these problems. Computational results are presented in § 7 and compared with experiments by Stewart and Townsend.

1. Introduction

During the past few years we have attempted† to apply to turbulence problems some ideas (Cameron & Martin 1947; Wiener 1958) concerning the

† See Meecham & Siegel 1964; Imamura, Meecham & Siegel 1965; Siegel, Imamura & Meecham 1965; the first two papers will be cited here as papers I and II.

representation of non-Gaussian non-linear processes. Saffman (1967) gives an interesting review of this and other recent theories.

The expansion used here is built around a Gaussian process, sometimes called the white noise process. Non-Gaussian characteristics are represented by polynomial functionals of the basic process. It is known that in some important ways developed turbulence is nearly Gaussian (for instance, in developed turbulence the lower-order moments of one component of the velocity are related very nearly as they would be for a Gaussian process). It is also known that some moments (notably of derivatives of the velocity field and more generally of velocity differences) show deviations from their Gaussian values. We shall argue here on the basis of experimental evidence that for many characteristics the non-Gaussian part is small compared with the Gaussian part of the developed process.

It is worth remarking that the formulation to be presented can yield only positive spectra (even for spectra of derivatives for instance) and can be shown to be a Galilean invariant. Calculations are given and compared with turbulence measurements.

2. Review of the Wiener representation

In past work we have dealt mainly with the application of the representation to Burgers' model equation. The general characteristics of the representation and its extension for the representation of vector processes have been treated previously in paper II. It is worth reviewing and specializing the representation here for application to problems involving the Navier–Stokes equation, and related problems.

The basic element in the representation is the white noise process. To gain some feeling for this basic element consider the following limiting operation. Begin with a scalar function, $H_\Delta^{(1)}(x)$, of a scalar variable defined as follows. Divide the x-axis into cells of width Δ. To achieve a representation of $H_\Delta^{(1)}$ select its value independently in each cell, from a Gaussian distribution of variance Δ^{-1}. Each member of the ensemble is a histogram; one member is shown in figure 1.

The idealized process derived from Wiener's work is obtained by allowing $\Delta \to 0$ for the above functional; the process so obtained will be designated $H^{(1)}(x)$. We shall need to represent vector fields and accordingly assign a statistically independent process of this type for each cartesian direction (at every point in space). The cells of figure 1 are replaced by volume elements Δ^3 and the variance becomes Δ^{-3}. Then we are led to consider $H^{(1)}(r)$ with the properties, following from the definitions,

$$\left.\begin{aligned} \langle H_i^{(1)}(\mathbf{r})\rangle &= 0, \\ \langle H_i^{(1)}(\mathbf{r}) H_j^{(1)}(\mathbf{r}')\rangle &= \delta_{ij}\delta(\mathbf{r}-\mathbf{r}'), \end{aligned}\right\} \tag{2.1}$$

with higher moments related as in a Gaussian distribution. A physical process can be represented by an integral

$$u_i^{(1)}(\mathbf{r}) = \int K_{i\alpha_1}^{(1)}(\mathbf{r}, \mathbf{r}_1) H_{\alpha_1}^{(1)}(\mathbf{r}_1)\, d\mathbf{r}_1 \tag{2.2}$$

with the repeated index summed. As in many turbulence theories, a more compact notation is desirable. Here we shall sometimes adopt a notation (see paper II)

$$R_1 \equiv \{\alpha_1, \mathbf{r}_1\}$$

and use a kind of generalized summation convention, writing for (2.2)

$$u_i^{(1)}(\mathbf{r}) = K_i^{(1)}(\mathbf{r}, R_1) H^{(1)}(R_1). \tag{2.3}$$

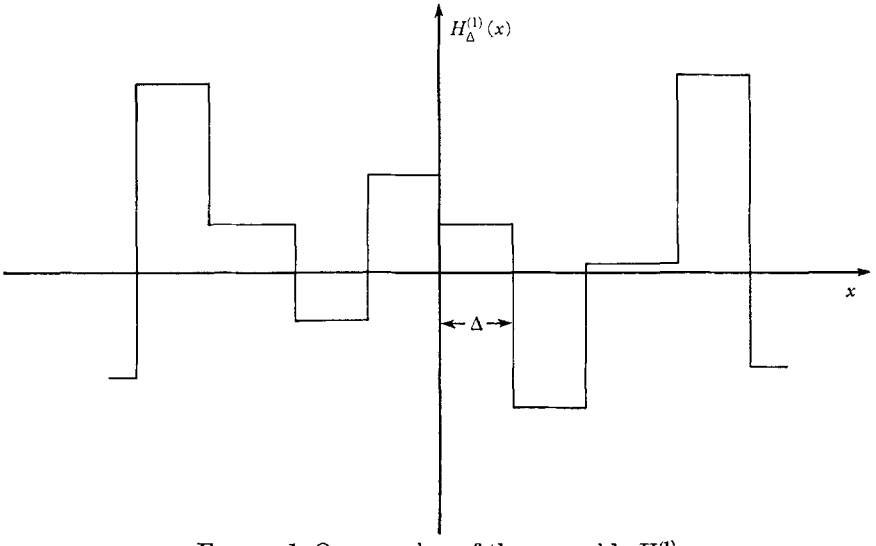

FIGURE 1. One member of the ensemble $H_\Delta^{(1)}$.

The repeated generalized variables R (and, below, S) in a single term are to be summed on α, and integrated over $\mathbf{r}$, as in (2.2). Likewise let $H^{(1)}(R_1)$ stand for $H_{\alpha_1}^{(1)}(\mathbf{r}_1)$, etc.

In (2.2) K is a non-random function. The process u_i, being a sum of independent Gaussian processes, is itself Gaussian. Further, almost every member of the ensemble of $\mathbf{u}$'s is continuous for regular K. The covariance of $\mathbf{u}$ is found, using (2.1),

$$\langle u_i(\mathbf{r}) u_j(\mathbf{r}') \rangle = \int K_{i\alpha}^{(1)}(\mathbf{r}, \mathbf{r}_1) K_{j\alpha}^{(1)}(\mathbf{r}', \mathbf{r}_1) d\mathbf{r}_1. \tag{2.4}$$

Evidently, even though the basic element of the representation is an idealized, irregular process, averages of physical processes can be quite regular.

A statistically homogeneous process can be represented by choosing a kernel which is a function of the difference of its two arguments. In such a case the covariance, for instance, becomes

$$\langle u_i(\mathbf{r}) u_j(\mathbf{r}') \rangle = \int U_{i\alpha}^{(1)}(\mathbf{r} - \mathbf{r}' + \mathbf{r}_1) U_{j\alpha}^{(1)}(\mathbf{r}_1) d\mathbf{r}_1. \tag{2.5}$$

We shall use the symbol U (in place of K) for homogeneous processes.

Consider the representation of non-Gaussian processes. For this purpose we define polynomial combinations of the white noise process in such a way that they are mutually statistically orthogonal (see paper II). This could be done in a

variety of ways, of course, but because of the Gaussian character of the basic process the most convenient functionals are based on the generalized Hermite polynomials. The first few are (including a zeroth-order, non-random function)

$$\left.\begin{aligned} H^{(0)}(\mathbf{r}) &= 1, \\ H^{(2)}(R_1, R_2) &= H^{(1)}(R_1)H^{(1)}(R_2) - \delta(R_1, R_2), \\ H^{(3)}(R_1, R_2, R_3) &= H^{(1)}(R_1)H^{(1)}(R_2)H^{(1)}(R_3) - H^{(1)}(R_1)\delta(R_2, R_3) \\ &\quad - H^{(1)}(R_2)\delta(R_3, R_1) - H^{(1)}(R_3)\delta(R_1, R_2) \end{aligned}\right\} \tag{2.6}$$

and similarly for higher orders. We define

$$\delta(R_1, R_2) = \delta_{\alpha_1\alpha_2}\delta(\mathbf{r}_1 - \mathbf{r}_2).$$

It is readily shown from the definitions that (on suppressing subscripts)

$$\langle H^{(m)}H^{(n)}\rangle = 0 \quad (m \neq n), \tag{2.7}$$

showing the statistical orthogonality mentioned above. The quantities defined in (2.6) are vector generalizations of the corresponding scalar quantities given in paper I, (2.4), and specializations of the results for the Wiener–Hermite functionals given in paper II. To represent a physical process, multiply the functionals of (2.6) by kernels and integrate in a way analogous to (2.2) [or (2.3)]. We have

$$u_i(\mathbf{r}) = \bar{u}_i(\mathbf{r}) + u_i^{(1)}(\mathbf{r}) + u_i^{(2)}(\mathbf{r}) + \ldots \tag{2.8}$$

with $\bar{u}_i$ the non-random, mean value of the velocity field and

$$u_i^{(1)}(\mathbf{r}) = K_i^{(1)}(\mathbf{r}, R_1)H^{(1)}(R_1), \tag{2.3}$$

$$u_i^{(2)}(\mathbf{r}) = K_i^{(2)}(\mathbf{r}, R_1, R_2)H^{(2)}(R_1, R_2), \tag{2.9}$$

$$u_i^{(3)}(\mathbf{r}) = K_i^{(3)}(\mathbf{r}, R_1, R_2, R_3)H^{(3)}(R_1, R_2, R_3).$$

$$\vdots$$

An important consequence of the definitions (2.6) can be seen. The higher-order functionals $H^{(2)}, H^{(3)}, \ldots$ are symmetric in all of their arguments, R_i. Consequently we can without loss of generality assume that $K^{(2)}, K^{(3)}, \ldots$ are similarly symmetric in the corresponding arguments ($\mathbf{r}$ excepted), e.g.

$$K^{(2)}_{i\alpha_1\alpha_2}(\mathbf{r}, \mathbf{r}_1, \mathbf{r}_2) = K^{(2)}_{i\alpha_1\alpha_2}(\mathbf{r}, \mathbf{r}_2, \mathbf{r}_1). \tag{2.10}$$

If the velocity field is known to be statistically homogeneous, we have

$$u_i^{(1)}(\mathbf{r}) = \int U^{(1)}_{i\alpha_1}(\mathbf{r} - \mathbf{r}_1)H^{(1)}_{\alpha_1}(\mathbf{r}_1)\,d\mathbf{r}_1,$$

$$u_i^{(2)}(\mathbf{r}) = \iint U^{(2)}_{i\alpha_1\alpha_2}(\mathbf{r} - \mathbf{r}_1, \mathbf{r} - \mathbf{r}_2)H^{(2)}_{\alpha_1\alpha_2}(\mathbf{r}_1, \mathbf{r}_2)\,d\mathbf{r}_1\,d\mathbf{r}_2. \tag{2.11}$$

(It does not seem desirable to continue the compact notation (2.3) for such problems.) From the type of symmetry in (2.10) it follows that $u^{(n)}$ are similarly symmetric, e.g. $U^{(2)}_{i\alpha_1\alpha_2}(\mathbf{r}_1, \mathbf{r}_2)$ is symmetric in subscripts 1 and 2.

The representation described above is appropriate to random initial value problems, of the type approximated by wind-tunnel, wake, and boundary-layer applications. For problems involving random forcing functions, the representation should be extended to include random characteristics in time

(in the above, time is an implicit parameter). For this purpose each point in space and time is to have an independent Gaussian process 'attached' with variance $\Delta^{-3}(\Delta t)^{-1}$ where Δt is the time increment corresponding to the space increment Δ.† The basic process is then $H_i^{(1)}(\mathbf{r},t)$ with properties corresponding to (2.1),

$$\left.\begin{aligned}\langle H_i^{(1)}(\mathbf{r},t)\rangle &= 0,\\ \langle H_i^{(1)}(\mathbf{r},t)H_j^{(1)}(\mathbf{r}',t')\rangle &= \delta_{ij}\delta(\mathbf{r}-\mathbf{r}')\delta(t-t').\end{aligned}\right\} \tag{2.12}$$

The process of (2.2) and (2.3) is generalized,

$$\begin{aligned}u_i^{(1)}(\mathbf{r},t) &= K_i^{(1)}(\mathbf{r},t;T_1,T_2)H^{(1)}(T_1,T_2)\\ &\equiv \iiiint K_{i\alpha_1}^{(1)}(\mathbf{r},t;\mathbf{r}_1,t_1)H_{\alpha_1}^{(1)}(\mathbf{r}_1,t_1)\,d\mathbf{r}_1\,dt_1\end{aligned} \tag{2.13}$$

and similarly for higher-order terms, with T_1 representing the set of quantities,

$$T_1 \equiv \{\alpha_1,\mathbf{r}_1,t_1\}, \quad \text{etc.} \tag{2.14}$$

The process can be made statistically stationary by assuming that time appears in $K^{(n)}$ as a difference variable in a way analogous to (2.11). In this paper we shall usually restrict attention to initial value problems, decaying turbulence.

3. Development of the dynamics using the functionals

We have up to now been considering the kinematics of processes. To determine the kernels we must of course use the dynamics, or more generally the equations governing the process. To continue, we substitute (2.8) in the equation of motion. For the first application of the theory consider incompressible viscous fluid flow. We use a (continuous) analogue of Kraichnan's (1959) formulation. Define the Fourier transform of the velocity field

$$u_i(\mathbf{k}) = \int e^{i\mathbf{k}.\mathbf{r}}\,u_i(\mathbf{r})\,d\mathbf{r} \tag{3.1}$$

and take the transform of the Navier–Stokes equation to find after manipulation

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)u_i(\mathbf{k}) = \tfrac{1}{2}i(2\pi)^{-3}P_{ijl}(\mathbf{k})\int u_j(\mathbf{k}-\mathbf{k}')\,u_l(\mathbf{k}')\,d\mathbf{k}'+f_i(\mathbf{k}) \tag{3.2}$$

with

$$P_{ijl}(\mathbf{k}) = k_l P_{ij}(\mathbf{k})+k_j P_{il}(\mathbf{k}), \tag{3.3}$$

$$P_{ij}(\mathbf{k}) = \delta_{ij}-k^{-2}k_i k_j. \tag{3.4}$$

Here and often below the dependence on time is implicit. Note that P_{ijl} and P_{jl} are symmetric in (j,l) and that

$$k_i P_{ij}(\mathbf{k}) = k_i P_{ijl}(\mathbf{k}) = 0. \tag{3.5}$$

In (3.2) f_i is the Fourier transform of the forcing term (per unit mass), if it is present. To continue this problem we represent $\mathbf{u}$ by its stochastic expansion (2.8) [and $\mathbf{f}$ by its corresponding series, with known kernels $F^{(n)}$ replacing the

† For simplicity we suppose that the random initial (and usually independent) transients have died out.

unknown $K^{(n)}$ of (2.3) and (2.9)]. We here specialize to a decaying process, $\mathbf{f} = 0$; the initial values of the field are assumed to be random. Development of the equations governing the various, needed kernels is now straightforward. The series (2.8) is substituted in (3.2). The quadratic, non-linear term gives products of functionals which can in turn be represented by series of the functionals themselves using results of paper II. Once the equation is written in terms of linear expressions in the functionals, the statistical orthogonality represented by (2.7) is used and the coefficients of the corresponding terms are equated.† We give the equations [except (3.6), which is more general] through terms of third order in the functionals; this is farther than the expansion need be carried for most applications. The equations may be applied to statistically inhomogeneous or anisotropic flows (for instance shear flows). It is emphasized that the kernels $K^{(n)}$ are symmetric in any interchange of their last n variables; it is important that initial values of the kernels also have this symmetry. We represent the transform of the mean flow by $\bar{u}_i(\mathbf{k})$ and find, for the force-free case ($\mathbf{f} = 0$),

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)\bar{u}_i(\mathbf{k}) = \tfrac{1}{2}i(2\pi)^{-3}P_{ijl}(\mathbf{k})\cdot\left\{\int \bar{u}_j(\mathbf{k}')\bar{u}_l(\mathbf{k}-\mathbf{k}')d\mathbf{k}' + \sum_{n=1}^{\infty} n!\int K_j^{(n)}(\mathbf{k}';R_1,\ldots,R_n)K_l^{(n)}(\mathbf{k}-\mathbf{k}';R_1,\ldots,R_n)d\mathbf{k}'\right\}. \quad (3.6)$$

In (3.6) we have included all the non-linear terms. The result follows easily from a use of the statistical orthogonality. If the process is laminar, the series vanishes, and we have (3.2). The $K_j^{(n)}(\mathbf{k};R_1\ldots)$ is the Fourier transform of $K_j^{(n)}(\mathbf{r},R_1\ldots)$. Continuing to higher orders we have

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)K_i^{(1)}(\mathbf{k};R_1) = i(2\pi)^{-3}P_{ijl}(\mathbf{k})\cdot\left\{\int d\mathbf{k}'[\bar{u}_j(\mathbf{k}')K_l^{(1)}(\mathbf{k}-\mathbf{k}';R_1) + 2K_j^{(1)}(\mathbf{k}';R_1')K_l^{(2)}(\mathbf{k}-\mathbf{k}';R_1,R_1') + 6K_j^{(2)}(\mathbf{k}';R_1',R_2')K_l^{(3)}(\mathbf{k}-\mathbf{k}';R_1,R_1',R_2')+\ldots]\right\}, \quad (3.7)$$

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)K_i^{(2)}(\mathbf{k};R_1,R_2) = \tfrac{1}{2}i(2\pi)^{-3}P_{ijl}(\mathbf{k})\cdot\left\{\int d\mathbf{k}'[2\bar{u}_j(\mathbf{k}')K_l^{(2)}(\mathbf{k}-\mathbf{k}';R_1,R_2) + K_j^{(1)}(\mathbf{k}';R_1)K_l^{(1)}(\mathbf{k}-\mathbf{k}';R_2) + 4K_j^{(2)}(\mathbf{k}';R_1,R')K_l^{(2)}(\mathbf{k}-\mathbf{k}';R_2,R') + 6K_j^{(1)}(\mathbf{k}';R')K_l^{(3)}(\mathbf{k}-\mathbf{k}';R_1,R_2,R')+\ldots]\right\}, \quad (3.8)$$

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)K_i^{(3)}(\mathbf{k};R_1,R_2,R_3) = i(2\pi)^{-3}P_{ijl}(\mathbf{k})\cdot\left\{\int d\mathbf{k}'[\bar{u}_j(\mathbf{k}')K_l^{(3)}(\mathbf{k}-\mathbf{k}';R_1,R_2,R_3) + \tfrac{1}{3}\Omega_{123}\{K_j^{(1)}(\mathbf{k}';R_1)K_l^{(2)}(\mathbf{k}-\mathbf{k}';R_2,R_3)\} + (3!/3)\Omega_{123}\{K_j^{(2)}(\mathbf{k}';R_1,R')K_l^{(3)}(\mathbf{k}-\mathbf{k}';R_2,R_3,R')\}]\right\}, \quad (3.9)$$

† For this result, it is necessary that the coefficient functions be symmetric in their arguments.

with $$\Omega_{123}\{F(R_1, R_2, R_3)\}$$
$$= F_{\alpha_1\alpha_2\alpha_3}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) + F_{\alpha_2\alpha_3\alpha_1}(\mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_1) + F_{\alpha_3\alpha_1\alpha_2}(\mathbf{r}_3, \mathbf{r}_1, \mathbf{r}_2). \quad (3.10)$$

An interesting consequence of the formulation used in paper II can be seen in the structure of these equations: the contribution to a given order of kernel of the non-linear (quadratic) term is analogous to the corresponding behaviour of Hermite polynomials. For instance, the contribution to the first-order term [right side of (3.7)] comes from the product of the zeroth with the first, the first with the second, the second with the third, etc. In general there will be a contribution to a given order if the given order lies between the sum and the difference of the orders on the right side and has the same parity as the sum (or difference). There would be similar, though more complicated, correspondences for higher-order (for instance, cubic) non-linearity.

Consider the effect of the requirement of incompressibility. The divergence of the expansion (2.8) must vanish for every member of the ensemble. If we use the statistical orthogonality of the functionals (2.7), together with the symmetry of the kernels $K^{(n)}$ in their last n arguments, we find that incompressibility leads to the requirement

$$k_\alpha \bar{u}_\alpha(k) = 0$$

and
$$k_\alpha K^{(n)}_\alpha(\mathbf{k}, R_1, \ldots, R_n) = 0 \quad (n \geqslant 1).$$

We specialize now to statistically homogeneous turbulence. To do this let the kernels be functions of the difference variables as in (2.11) and let $\bar{u} = 0$. It is convenient to define the multiple transform of the kernels $U^{(n)}$

$$U^{(n)}_{j\alpha_1\ldots\alpha_n}(\mathbf{k}_1, \ldots, \mathbf{k}_n) = \int \ldots \int_n \exp[i(\mathbf{k}_1 . \mathbf{r}_1 + \ldots + \mathbf{k}_n . \mathbf{r}_n)]$$
$$\times U^{(n)}_{j\alpha_1\ldots\alpha_n}(\mathbf{r}_1, \ldots, \mathbf{r}_n) d\mathbf{r}_1 \ldots d\mathbf{r}_n. \quad (3.11)$$

The relation to the previous kernels is

$$K^{(n)}_j(\mathbf{k}; R_1, \ldots, R_n) = (2\pi)^{-3n+3} \int \ldots \int_n \delta(\mathbf{k} - \mathbf{k}_1 - \ldots - \mathbf{k}_n)$$
$$\times \exp[i(\mathbf{k}_1 . \mathbf{r}_1 + \ldots + \mathbf{k}_n . \mathbf{r}_n)] U^{(n)}_{j\alpha_1\ldots\alpha_n}(\mathbf{k}_1, \ldots, \mathbf{k}_n) d\mathbf{k}_1 \ldots d\mathbf{k}_n. \quad (3.12)$$

We know the kernels $K^{(n)}$ are symmetric under any interchange of their last n variables, $R_1, R_2, \ldots$. From (3.12), taking the $(3n)$th order inverse Fourier transform, we see that $U^{(n)}$ are symmetric in the interchange of any pair of $\mathbf{k}_1, \alpha_1$; $\mathbf{k}_2, \alpha_2$; etc. Also from (3.11) we see $U^{(n)}$ has the same symmetry in $\mathbf{r}$'s. Again it is important that initial values of $U^{(n)}$ likewise have this symmetry. The equations for statistically homogeneous processes (through terms of third order) are obtained by substituting (3.12) in (3.7)–(3.9) and simplifying,

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) U^{(1)}_{i\alpha}(\mathbf{k}) = 2i(2\pi)^{-3} P_{i\gamma\delta}(\mathbf{k}) \int U^{(1)}_{\gamma\beta}(\mathbf{k}') U^{(2)}_{\delta\alpha\beta}(\mathbf{k}, -\mathbf{k}') d\mathbf{k}'$$
$$+ \left\{ 6i(2\pi)^{-6} P_{i\gamma\delta}(\mathbf{k}) . \iint U^{(2)}_{\gamma\alpha_1\alpha_2}(\mathbf{k}_1, \mathbf{k}_2) U^{(3)}_{\delta\alpha\alpha_1\alpha_2}(\mathbf{k}, -\mathbf{k}_1, -\mathbf{k}_2) d\mathbf{k}_1 d\mathbf{k}_2 \right\}, \quad (3.13)$$

$$\left(\frac{\partial}{\partial t}+\nu|\mathbf{k}_1+\mathbf{k}_2|^2\right) U^{(2)}_{i\alpha_1\alpha_2}(\mathbf{k}_1,\mathbf{k}_2) = \tfrac{1}{2}iP_{i\gamma\delta}(\mathbf{k}_1+\mathbf{k}_2)U^{(1)}_{\gamma\alpha_1}(\mathbf{k}_1)U^{(1)}_{\delta\alpha_2}(\mathbf{k}_2)$$
$$+\left\{i(2\pi)^{-3}P_{i\gamma\delta}(\mathbf{k}_1+\mathbf{k}_2)\left[2\int U^{(2)}_{\gamma\alpha_1\epsilon}(\mathbf{k}_1,\mathbf{k}')U^{(2)}_{\delta\alpha_2\epsilon}(\mathbf{k}_2,-\mathbf{k}')d\mathbf{k}'\right.\right.$$
$$\left.\left.+3\int U^{(1)}_{\gamma\epsilon}(\mathbf{k}')U^{(3)}_{\delta\alpha_1\alpha_2\epsilon}(\mathbf{k}_1,\mathbf{k}_2,-\mathbf{k}')d\mathbf{k}'\right]\right\}, \tag{3.14}$$

$$((\partial/\partial t)+\nu|\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3|^2)U^{(3)}_{i\alpha_1\alpha_2\alpha_3}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = iP_{ijl}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)$$
$$\times\left[\tfrac{1}{3}\Omega_{123}U^{(1)}_{j\alpha_1}(\mathbf{k}_1)U^{(2)}_{l\alpha_2\alpha_3}(\mathbf{k}_2,\mathbf{k}_3)+2(2\pi)^{-3}\Omega_{123}\int U^{(2)}_{j\alpha_1\beta}(\mathbf{k}_1,\mathbf{k}')U^{(3)}_{l\alpha_2\alpha_3\beta}(\mathbf{k}_2,\mathbf{k}_3,-\mathbf{k}')dk'\right]. \tag{3.15}$$

Let (3.13S) and (3.14S) stand for the simplified, lowest-order equations obtained by dropping terms in the brackets { } in (3.13) and (3.14) above.

The kernels $U^{(n)}$ above satisfy the reality condition (they are the transforms of real functions), $n \geqslant 1$,

$$U^{(n)*}_{i\alpha_1\ldots\alpha_n}(\mathbf{k}_1,\ldots,\mathbf{k}_n) = U^{(n)}_{i\alpha_1\ldots\alpha_n}(-\mathbf{k}_1,\ldots,-\mathbf{k}_n) \tag{3.16}$$

and the incompressibility condition

$$(\mathbf{k}_1+\ldots+\mathbf{k}_n)_\alpha U^{(n)}_{\alpha\alpha_1\ldots\alpha_n}(\mathbf{k}_1,\ldots,\mathbf{k}_n) = 0. \tag{3.17}$$

Consider the consequences of assuming that the flow is substantially statistically isotropic. The $U^{(n)}$ are isotropic tensors (see Batchelor 1953, pp. 42–3). Using this property and the reality condition (3.16) we find $U^{(n)}$ are pure real or pure imaginary according to whether n is odd or even.

The combination of the requirements that the flow be statistically homogeneous and isotropic as well as incompressible gives

$$U^{(1)}_{ij}(\mathbf{k}) = U_1(k)\,P_{ij}(k), \tag{3.18}$$

$$U^{(2)}_{i\alpha_1\alpha_2}(\mathbf{k}^{(1)},\mathbf{k}^{(2)}) = iP_{i\alpha}(\mathbf{k}^{(1)}+\mathbf{k}^{(2)})\,.\{V_1(k^{(1)}_\alpha k^{(1)}_{\alpha_1}k^{(1)}_{\alpha_2}+k^{(2)}_\alpha k^{(2)}_{\alpha_1}k^{(2)}_{\alpha_2})$$
$$+V_2(k^{(1)}_{\alpha_1}\delta_{\alpha\alpha_2}+k^{(2)}_{\alpha_2}\delta_{\alpha\alpha_1})+V_3(k^{(1)}_{\alpha_2}\delta_{\alpha\alpha_1}+k^{(2)}_{\alpha_1}\delta_{\alpha\alpha_2})+V_4(k^{(2)}_\alpha k^{(1)}_{\alpha_1}k^{(1)}_{\alpha_2}+k^{(1)}_\alpha k^{(2)}_{\alpha_1}k^{(2)}_{\alpha_2})\}, \tag{3.19}$$

with U_1 a real function and V_i real functions [of t; $(\mathbf{k}^{(1)})^2$, $(\mathbf{k}^{(2)})^2$ and $\mathbf{k}^{(1)}.\mathbf{k}^{(2)}$]. There are similar, more complicated forms for the real function $U^{(3)}_{i\alpha_1\alpha_2\alpha_3}$.

We summarize the results so far and compare with some previous work. For shear flows and other statistically inhomogeneous flows consider (3.6)–(3.9). We can proceed in two ways. First, to emphasize connexions with previous efforts, consider a perturbation result. Sum (3.7), (3.8), etc., retaining terms of lowest order in the kernels. Let u'_i represent the sum of the fluctuations so (2.8) becomes

$$u_i(\mathbf{r}) = \overline{u}_i(\mathbf{r})+u'_i(\mathbf{r}). \tag{3.20}$$

We have, for (3.6),

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)\overline{u}_i(\mathbf{k}) = \tfrac{1}{2}i(2\pi)^{-3}P_{ijl}(\mathbf{k})\left\{\int d\mathbf{k}'[\overline{u}_j(\mathbf{k}')\,\overline{u}_l(\mathbf{k}-\mathbf{k}')+u'_j(\mathbf{k}')\,u'_l(\mathbf{k}-\mathbf{k}')]\right\} \tag{3.21}$$

and

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)u'_i(\mathbf{k}) = i(2\pi)^{-3}P_{ijl}(\mathbf{k})\int\overline{u}_j(\mathbf{k}-\mathbf{k}')\,u'_l(\mathbf{k}')\,d\mathbf{k}', \tag{3.22}$$

where by incompressibility $k_\alpha \overline{u}_\alpha = k_\alpha u'_\alpha = 0$. Equations (3.20)–(3.22) are (the Fourier transform of) the usual perturbation equations. The last term of (3.21) is the Reynolds stress.

In the spirit of the present work, we might in a simple treatment first assume that the fluctuation is nearly Gaussian. Experiments support the view that at least certain characteristics of the flow have this property (see Townsend 1956, §7.6). Then let $K^{(n)} = 0, n \geqslant 2$. For (3.6), replace in (3.21),

$$u'_j(\mathbf{k}')\, u'_l(\mathbf{k}-\mathbf{k}') \Rightarrow \int K^{(1)}_{j\alpha}(\mathbf{k}';\mathbf{r}_1)\, K^{(1)}_{l\alpha}(\mathbf{k}-\mathbf{k}';\mathbf{r}_1)\, d\mathbf{r}_1; \tag{3.23}$$

for (3.7), replace in (3.22),

$$u'_i(\mathbf{k}) \Rightarrow K^{(1)}_{i\alpha}(\mathbf{k};\mathbf{r}_1), \tag{3.24}$$

where $K^{(1)}$ satisfies the incompressibility condition

$$k_i K^{(1)}_{i\alpha}(\mathbf{k};\mathbf{r}_1) = 0. \tag{3.25}$$

Inhomogeneous flows can sometimes be regarded as locally isotropic. This provides a further simplification. Define the full transform of $K^{(1)}$,

$$K^{(1)}_{j\alpha}(\mathbf{k}';\mathbf{k}'') \equiv \int e^{i\mathbf{k}''\cdot\mathbf{r}''} K^{(1)}_{j\alpha}(\mathbf{k}';\mathbf{r}'')\, d\mathbf{r}''. \tag{3.26}$$

This quantity becomes, using local isotropy and (3.25) (see Batchelor 1953, p. 43),

$$K^{(1)}_{j\alpha}(\mathbf{k}';\mathbf{k}'') = P_{j\beta}(\mathbf{k}')\{A_1 k''_\beta k''_\alpha + A_2 \delta_{\beta\alpha} + A_3 k''_\beta k'_\alpha\}; \tag{3.27}$$

here

$$A_i = A_i(\mathbf{k}'^2, \mathbf{k}''^2, \mathbf{k}'.\mathbf{k}''; t).$$

Proudman & Reid (1954) have suggested that incompressibility is conveniently introduced by multiplying by P_{tj} as here. Further symmetry assumptions for special geometries are possible (circular jets, plane boundary layers, etc.). By (3.27) and the reality condition we see that A_i are real.

We collect results (and simplify) for statistically homogeneous and isotropic flows, (3.13)–(3.15). We suppose again that the process is nearly Gaussian (discussed in the next section), retaining $U^{(1)}$ and $U^{(2)}$ in (3.13S) and (3.14S). Integrate the latter to find (time now being shown explicitly)

$$\begin{aligned} U^{(2)}_{i\alpha_1\alpha_2}(\mathbf{k}_1,\mathbf{k}_2;t) = {} & U^{(2)}_{i\alpha_1\alpha_2}(\mathbf{k}_1,\mathbf{k}_2;0)\exp\{-\nu(\mathbf{k}_1+\mathbf{k}_2)^2 t\} \\ & + \tfrac{1}{2} i P_{i\gamma\delta}(\mathbf{k}_1+\mathbf{k}_2)\, P_{\gamma\alpha_1}(\mathbf{k}_1)\, P_{\delta\alpha_2}(\mathbf{k}_2) \\ & \times \int_0^t U_1(k_1,t')\, U_1(k_2,t') \exp\{-\nu(\mathbf{k}_1+\mathbf{k}_2)^2 (t-t')\}\, dt'. \end{aligned} \tag{3.28}$$

We have used the isotropic form (3.18). The function U_1 is real; the initial value of $U^{(2)}$ is pure imaginary and is symmetric under the interchange of the subscripts 1 and 2. It is assumed that the process is initially isotropic. The third-rank tensor, $U^{(2)}$, can be written in terms of the four generating scalars as in (3.19). Furthermore, if the initial value of the second-order term is negligible, the term can be represented by a single scalar function; the tensor is of the form of the second term on the right side of (3.28).

Use (3.18) and sum (3.13S) on its free indices to find

$$\left(\frac{\partial}{\partial t}+\nu k^2\right) U_1(k,t) = i(2\pi)^{-3}P_{\alpha jl}(\mathbf{k})\int U_1(k',t)P_{j\beta}(\mathbf{k}')U^{(2)}_{l\alpha\beta}(\mathbf{k},-\mathbf{k}';t)d\mathbf{k}'. \quad (3.29)$$

The equations (3.28) and (3.29) form the system appropriate, to lowest significant order, to the discussion of statistically homogeneous and isotropic turbulence in incompressible fluid. The various moments (correlations), skewness factors, etc., will be formed from $U^{(1)}$ and $U^{(2)}$ in a way analogous to (2.5).

4. Extent of Gaussianity from experiments

Consider statistically homogeneous and isotropic processes. First, we express the usual quantities of turbulence theory in terms of the kernels in the Wiener representation. For this purpose we need statistical expectations for special products of functionals [the general formula is given in paper II]. Then (see (2.1))

$$\langle H^{(1)}(R_1)H^{(1)}(R_2)\rangle = \langle H^{(1)}_{i_1}(\mathbf{r}_1)H^{(1)}_{i_2}(\mathbf{r}_2)\rangle \quad (4.1)$$

$$= \delta(R_1,R_2), \quad (4.2)$$

$$\langle H^{(1)}(R_1)H^{(1)}(R_2)H^{(2)}(R_3,R_4)\rangle = \delta(R_1,R_3)\delta(R_2,R_4)+\delta(R_1,R_4)\delta(R_2,R_3). \quad (4.3)$$

From (4.2) and (4.3) and the usual definitions (Batchelor 1953) we have for the energy spectrum function

$$E(k) = E_1(k)+E_2(k), \quad (4.4)$$

$$E_1(k) = 2(2\pi)^{-2}k^2U_1^2(k), \quad (4.5)$$

$$E_2(k) = 2(2\pi)^{-5}k^2\int U^{(2)}_{\gamma\alpha\beta}(\mathbf{k}',\mathbf{k}-\mathbf{k}')U^{(2)*}_{\gamma\alpha\beta}(\mathbf{k}',\mathbf{k}-\mathbf{k}')d\mathbf{k}'. \quad (4.6)$$

U_1 and $U^{(2)}$ are obtained by integrating (3.28) and (3.29). The longitudinal velocity correlation ($\hat{\imath}$ a unit vector in the x-direction)

$$Q(r) \equiv \langle u_1(\mathbf{x})u_1(\mathbf{x}+r\hat{\imath})\rangle \quad (4.7)$$

is found from the relation

$$Q(r) = 2\int_0^\infty E(k)k^{-2}r^{-2}\left(\frac{\sin(kr)}{kr}-\cos(kr)\right)dk. \quad (4.8)$$

The Gaussian contribution to the correlation is, using (4.5),

$$Q(r) = \frac{r^{-2}}{\pi^2}\int_0^\infty U_1^2(k)\left(\frac{\sin(kr)}{kr}-\cos(kr)\right)dk. \quad (4.9)$$

The Gaussian part of the correlation tensor is also given in other form by (2.5).† The (lowest order) non-Gaussian part of the longitudinal correlation is found by substituting (4.6) for E in (4.8). A useful relation is found by manipulating the definitions,

$$r^{-2}\frac{d}{dr}(r^3Q_{1,2}) = \int e^{i\mathbf{k}\cdot\mathbf{r}}\frac{E_{12}(k)}{2\pi k^2}d\mathbf{k}.$$

† We shall call E_1 and E_2 the Gaussian and non-Gaussian energies, respectively. It should be emphasized that a representation can be chosen which gives a non-zero E_2, even for a Gaussian process. This can be done through the use of a 'measure-preserving transformation' (see Wiener 1958).

An interesting consequence of these formulae [see (4.5) and (4.6)] is that the energy spectra are necessarily positive. This is a general property of the theory presented here, and applies to all other spectra, for instance the spectra of derivatives of the velocity field.

Consider third-order velocity correlations. We write the usual definition

$$S_{ij,k}(\mathbf{r}) = \langle u_i(\mathbf{x})\, u_j(\mathbf{x})\, u_k(\mathbf{x}+\mathbf{r})\rangle, \tag{4.10}$$

then use (4.3) and the definition of the kernels $U^{(n)}$ to find to lowest order (the purely Gaussian term vanishes, of course)

$$S_{ij,k}(\mathbf{r}) = 2\iint \{U^{(1)}_{i\alpha}(\mathbf{r}')U^{(1)}_{j\beta}(\mathbf{r}'')U^{(2)}_{k\alpha\beta}(\mathbf{r}+\mathbf{r}', \mathbf{r}+\mathbf{r}'') + U^{(1)}_{k\alpha}(\mathbf{r}+\mathbf{r}')[U^{(2)}_{i\beta}(\mathbf{r}'')U^{(2)}_{j\alpha\beta}(\mathbf{r}', \mathbf{r}'') + U^{(1)}_{j\beta}(\mathbf{r}'')U^{(2)}_{i\alpha\beta}(\mathbf{r}', \mathbf{r}'')]\}\, d\mathbf{r}'\, d\mathbf{r}''. \tag{4.11}$$

This triple correlation is closely related to the transfer term in the energy equation as usually formulated. We shall discuss this formulation in more detail in the next section when we examine the zero-fourth-cumulant approximation.

It is possible to calculate other characteristics to the lowest order using U_1 and $U^{(2)}$. In particular, for the present purpose, consider the dimensionless flatness factor. One finds

$$\begin{aligned} \text{F.F.} &\equiv \langle u_1^4\rangle / \langle u_1^2\rangle^2 \\ &\equiv 3 + O(U^{(2)}/U_1)^2. \end{aligned} \tag{4.12}$$

The deviation is second order in the non-Gaussian part. This result can be greatly extended. It is seen that odd moments of isotropic processes vanish for $\mathbf{r} \to 0$. Such quantities are called one-point tensors and must be made up of Kronecker delta functions. There is no way to do this for odd-ranked tensors. Even-ranked tensors are of second order in the non-Gaussian contribution [see paper II]. We have the conclusion that the distribution at one point of a nearly normal process is normal *to terms of second order*. The result uses isotropy; deviations from such a rule can indicate anisotropy. Note that the deviation of (4.12) from 3·0 would be proportional to the third-order functional. Thus, if the flatness factor is *very* near to 3, the indication is that the term cubic in the Gaussian white noise process (the third functional) is very small compared with the quadratic term (the second functional). Present experimental evidence is not really conclusive, but tends to indicate such a behaviour.

Let us continue in more detail with a discussion of the relation to experiments of this theoretical statement of the turbulence problem. The evidence in support of the view that turbulence is in some important ways nearly normal has been given by, for instance, Proudman & Reid (1954). Similarly it is well known that some characteristics exhibit non-Gaussian behaviour. In general these latter involve differences, or derivatives, of field functions taken at two points which are close together compared with the scale of the turbulence. The most recent and complete experiments of this kind are those of Frenkiel & Klebanoff (1965). It is reasonable to conclude from this evidence that the larger-scale characteristics are substantially normal, whereas for the smallest scales there is deviation from normality. Even in this range there remains the possibility that the substantial

non-Gaussianity is due to intermittency (see Batchelor 1953); i.e. that the small-scale process has non-zero amplitudes in restricted regions only and is essentially Gaussian there. This raises the possibility that the nearly normal treatment may be valid, if adjusted, for the very small-scale process. Our problem is to try, insofar as is at present possible, to determine the nature of the deviations from Gaussianity. For this purpose it is noted, see (4.11), that the triple velocity

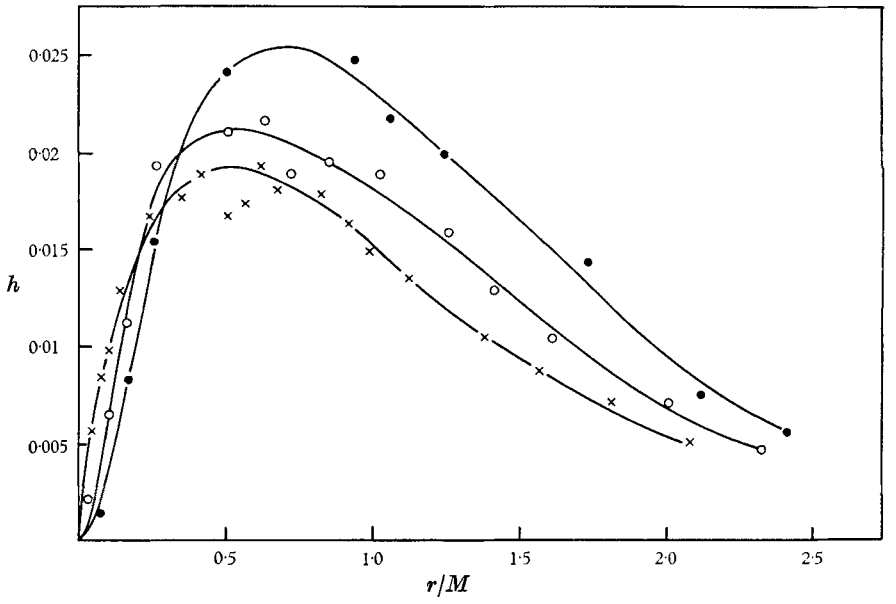

FIGURE 2. $x/M = 30$. ●, $U = 620$ cm/sec; $M = 1{\cdot}27$ cm; $R_M = 5300$. ○, $U = 620$ cm/sec; $M = 5{\cdot}08$ cm; $R_M = 21{,}200$. ×, $U = 1240$ cm/sec; $M = 5{\cdot}08$ cm; $R_M = 42{,}400$. (After Stewart 1951.)

correlation or the energy transfer term is well suited. This correlation is linear in the non-Gaussian part of the process, to lowest order [see (4.11)]. Consider the measurements by Stewart (1951) of

$$h(r) = -\tfrac{1}{2}\langle u_1^2 u_1' \rangle . \langle u^2 \rangle^{-\frac{3}{2}}. \tag{4.13}$$

First, for a proper criterion one might remove the factor $\frac{1}{2}$ and normalize with $\langle u_1^2 \rangle$; on the other hand the triple correlation involves three terms (4.11) with $K^{(2)}$. We shall accordingly take for a criterion the heuristic relation

$$h(r) \sim |K^{(2)}/K^{(1)}| \tag{4.14}$$

in dimensionless form, using the correlation length and the r.m.s. velocity fluctuation for scaling. The results obtained by Stewart for different Reynolds numbers are shown in figure 2.

The most important point in the present context is that the non-Gaussian part of the velocity process is between $1\frac{1}{2}$ and $2\frac{1}{2}$ % of the Gaussian part by (4.14); certainly small enough to encourage the use of a nearly normal approximation. (It is of course true that the triple correlation goes to zero at the origin for symmetry reasons, and consequently small-scale non-Gaussian characteristics

are somewhat suppressed.) An interesting sidelight is that, as the Reynolds number increases, the relative non-Gaussian part of the process in fact *decreases*. This is perhaps to be expected if we believe that the process is very nearly Gaussian when it is generated at the grid and that, the more violent the process of generation, the more random the initial impulses. This remark has interest in the following context. It is sometimes thought that the flow is quite non-Gaussian at generation, and becomes Gaussian through the action of a (modified) central limit theorem even before final decay. From the viewpoint of the present theory and its relation to experiments another possibility is suggested. The process begins very nearly Gaussian and (we shall see) becomes somewhat less Gaussian as we proceed downstream. It should be remarked that if the process is thought to be initially quite non-Gaussian then the adjustment time is embarrassingly short, of order one characteristic time of the process, as can be seen in figure 3.

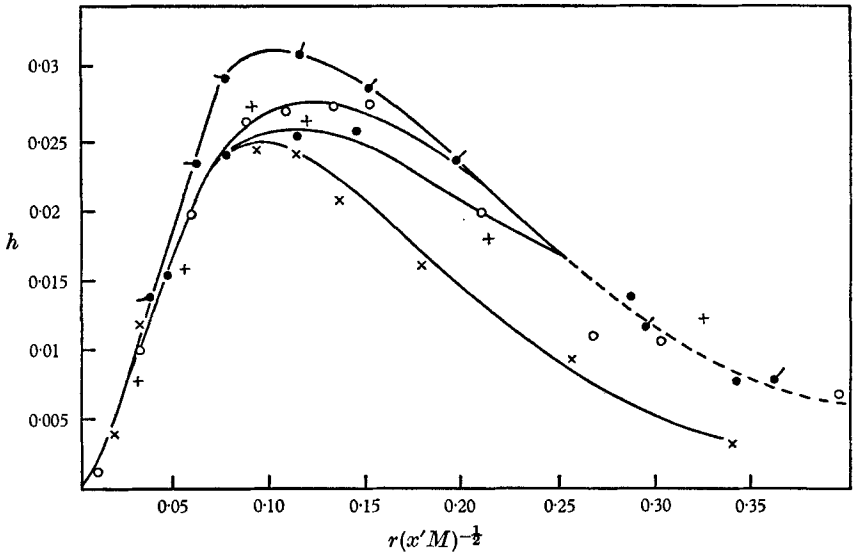

FIGURE 3. $R_M = 5300$. ×, $x/M = 20$; ●, $x/M = 20$; ○, $x/M = 60$; +, $x/M = 90$; ♦, $x/M = 120$; $x' = x + 5M$ (After Stewart 1951.)

To interpret these results in terms of dimensionless decay time, divide x/M by 20 (assuming the velocity fluctuation is 5% of the mean flow).

Consider an interesting sidelight concerning the stochastic truncation discussed in this paper. In order to integrate the equations of motion [(3.13*S*) and (3.14*S*) for instance] we must be given, in principle at least, initial values for the first two kernels. These initial values can be determined for the first two kernels (first two terms in the expansion) from initial values of, for instance, the two-point and the three-point velocity correlations. Then all statistical characteristics for later times are determined in terms of just these two (initial) correlations. Higher-order truncations would involve successively higher-order initial correlations. Or, on the other hand, use of just the Gaussian term would imply that all characteristics were determined for all time by the initial value of the simple, second-order velocity correlation.

5. Discussion of the zero-fourth-cumulant approximation

We begin by describing the use of the approximation following Proudman & Reid (1954). Suppose the Navier–Stokes equation (3.2), force-free, is multiplied by the velocity field at a second point and averaged. One obtains an equation connecting the second-order velocity correlation (2.5) with a third-order correlation (4.10) on the right side. Then the process is modified and repeated: the Navier–Stokes equation is multiplied by two field functions at two different points and averaged. We obtain an equation connecting the third-order correlation with the fourth-order correlation, the average of the product of four velocity fields. Finally, the fourth-order correlation is approximated using the value which the correlation would have if the velocity field were normally distributed. The result of these manipulations is

$$[(\partial/\partial t)+2\nu k^2]E(k) = T(k) \tag{5.1}$$

(the energy equation, where T is the transfer term); T is given by a linear, integral operation on Φ and Ψ and

$$\left[\frac{\partial}{\partial t}+\nu(k^2+k'^2+k''^2)\right]\Phi = \frac{E_1(k'')}{16\pi^2 k''^2}\left[\frac{E_1(k')}{k'^2}-\frac{E_1(k'')}{k''^2}\right], \tag{5.2}$$

$$[(\partial/\partial t)+\nu(k^2+k'^2+k''^2)]\Psi = 0, \tag{5.3}$$

with $\mathbf{k}+\mathbf{k}'+\mathbf{k}'' = 0$.

The general nature of the zero-fourth-cumulant approximation can be seen from (5.1)–(5.3). The effect of the non-linear term on the transfer process is given by the right-hand side of (5.2). It has been set equal to what it would have been if the velocity field were Gaussian, and thus becomes quadratic in the energy spectrum function. We have emphasized the Gaussian nature of the transfer with the subscript 1 [see (4.5)]; the energy spectrum function appearing in this role in the equations is the Gaussian part. It is true that non-Gaussian processes can have the zero-fourth-cumulant characteristic seen in (5.2) but in such a case one would have to fix the quasi-normal behaviour as a side condition.

Consider now the energy equation obtained from the Wiener expansion. This can be obtained in a number of ways, all equivalent to lowest order. If the process outlined for the zero-fourth-cumulant is followed step by step using the stochastic expansion and if the non-linear transfer in (5.2) is approximated by dropping all but the first term, U_1, in the expansion, then precisely the equations (5.1)–(5.3) are obtained with the understanding that in the left-hand side $E = E_1 + E_2$, the sum of the Gaussian and the lowest-order (quadratic) non-Gaussian contributions to the energy spectrum. In the right-hand side we have only the Gaussian term, E_1. In such a case, the system is completed by an extra equation for E_2, in effect the equation (3.28). The result of integrating these equations is an energy spectrum function, E, which is, from the structure, necessarily positive.

The zero-fourth-cumulant approximation has an important flaw, as can be seen from this discussion. It is essential in a proper treatment to include the time rate of change of the non-Gaussian part of E in the energy equation. This is not done in the usual treatment of (5.1)–(5.3), where a single energy spectrum func-

tion is used for E and E_1. The fact is that the *time rates of change* of the Gaussian term and the lowest-order non-Gaussian term are of the same order, even though for moderate times the latter is much smaller than the former. To see this consider a slightly different method for the derivation of the energy equation, using (3.13S) and (3.14S). Construct the energy spectrum function (4.4)–(4.6) by multiplying (3.13S) by $U^{(1)}$ and adding to it the result of integrating the product of (3.14S) and $U^{(2)}$. It is found in this way that $\partial E_1/\partial t$ and $\partial E_2/\partial t$ are of the same order. Indeed it can be shown, because of energy conservation, that the integral of these two expressions, except for the viscous terms, vanishes; their integrals are equal and opposite.

The flaw in the formulation of the zero-fourth-cumulant approximation just discussed is not, however, responsible for the poor results which have been obtained from computations (O'Brien & Francis 1962; Ogura 1963). There is a more basic difficulty with these calculations which we now consider. A necessary condition for the validity of the zero-fourth-cumulant approximation is that the non-Gaussian term remain small compared with the Gaussian term in the random process expansion. As outlined in § 4, a convenient way to estimate the relative effects of these terms is to form the dimensionless triple correlation, or equivalently the dimensionless transfer term. Alternatively, we can use the dimensionless time rate of change of the energy spectrum function [see (5.1)]. Now consider the zero-fourth-cumulant calculations. The energy spectrum function at $t = 0$, examined by Proudman & Reid (1954), is

$$E(k,0) = E_0 x^4 e^{-x^2}, \quad x = k/k_0, \tag{5.4}$$

where k_0 established the initial length scale. For this initial value, after $t \approx 1$, T can be estimated to be less than about 0·2 (see $\partial T/\partial t$ plotted in their figure 2), assuming moderate changes in T during this time. Consider Ogura's (1963) calculation for later times. First, in the present units, Ogura's time, c, should be divided by about 20. Then, for that author's largest Reynolds number, $R_\lambda = 28{\cdot}8$ (square this to find the approximate fluctuation Reynolds number), it is seen that $\partial E/\partial t \sim 1$, see his figure 1. Clearly the nearly normal assumption is violated.

It is interesting that energy spectra as observed for real fluids give much smaller transfers. There is a suggestion that for the present theory computations show the transfers remain small for larger times. Indeed, the experiments show (see figure 3) that the triple correlation, proportional to the transfer, remains very small to $t = 6$, in units of the correlation time. The suggestion is that, if realistic initial values for the spectrum function are used, a calculation based on the present theory should be valid for a considerable time, probably to ten correlation times and beyond.

An interesting sidelight is that there is a considerable amount of evidence showing that to lowest order the present expansion is more stable for computation than is the zero-fourth-cumulant approximation. For instance, for Burgers' model equation calculations discussed in paper I† it is found that the energy

† The approach used in paper I is the scalar equivalent of (3.13S) and (3·14S). However, in the latter equation the second-order kernel is approximated by setting it equal to the right-hand side; this is an acceptable approximation for small changes (moderate times).

spectrum achieves equilibrium at $t = 0{\cdot}1$ (time measured in units of the correlation time) and holds its form at least to $t = 3{\cdot}0$, where the calculation was ended. The corresponding quasi-normal calculation (Jeng, Foerster, Haaland & Meecham 1966) for most initial values yielded negative energies and subsequently became unstable for about $t = 1$. Calculation shows that the present expansion for the Navier–Stokes equation gives stable spectra at least up to $t = 6$, for $R' = 250$. For similar R', Ogura (1963) found negative spectra at $t = 2{\cdot}5$ and poor behaviour (oscillation) before that. The improvement in stability is presumably due to the distinction between E and E_1 discussed above.

In §7 we apply the theory developed in previous sections to turbulent flow experiments.

6. Convective transport of a reactant

Consider the turbulent mixing of a dynamically passive reactant, with a possible first-order reaction. We shall refer to the formulation of this problem given in paper III (O'Brien & Francis 1962). If Γ is the concentration of the reactant we have for its conservation equation

$$\left(\frac{\partial}{\partial t} - D\nabla^2 + C\right)\Gamma \equiv \mathscr{L}\Gamma = -\frac{\partial}{\partial r_j} u_j \Gamma. \tag{6.1}$$

The coefficients of fluid viscosity, diffusion (D), and the reaction rate constant (C) are assumed constant. Since the mixing is dynamically passive, $\mathbf{u}$ is assumed known together with its statistics. In application $\mathbf{u}$ would be determined as described in §§ 1–4. For an example, Γ could be the temperature in a suitable thermal-mixing problem. We formulate a general process which may be inhomogeneous and anisotropic.

First Γ is statistically dependent on $\mathbf{u}$ as a result of mixing. Following the representation above let $\mathbf{u}$ be given by (2.8), (2.2) and (2.9). Then Γ is similarly represented, remembering its scalar character, with time implicit:

$$\Gamma(\mathbf{r}) = \bar{\Gamma}(\mathbf{r}) + \Gamma^{(1)}(\mathbf{r}) + \Gamma^{(2)}(\mathbf{r}) + \ldots. \tag{6.2}$$

$\bar{\Gamma}$ is the average value of Γ and

$$\begin{aligned} \Gamma^{(1)}(\mathbf{r}) &= \int G^{(1)}_{\alpha_1}(\mathbf{r}; \mathbf{r}_1) H^{(1)}_{\alpha_1}(\mathbf{r}_1)\, d\mathbf{r}_1 \\ &\equiv G^{(1)}(\mathbf{r}; R_1) H^{(1)}(R_1), \\ \Gamma^{(2)}(\mathbf{r}) &= \iint G^{(2)}_{\alpha_1\alpha_2}(\mathbf{r}; \mathbf{r}_1, \mathbf{r}_2) H^{(2)}_{\alpha_1\alpha_2}(\mathbf{r}_1, \mathbf{r}_2)\, d\mathbf{r}_1 d\mathbf{r}_2 \\ &\equiv G^{(2)}(\mathbf{r}; R_1, R_2) H^{(2)}(R_1, R_2). \end{aligned} \tag{6.3}$$

As before, sum on R_i when repreated in the same term. Equations (2.8) and (6.2) are substituted in (6.1). The quadratic, non-linear term is re-expressed using results of paper II. In the resulting expression, linear in the functionals, equate

coefficients. We find through (at least) terms linear in the second-order functionals, $K^{(2)}$ and $G^{(2)}$,†

$$\mathscr{L}\bar{\Gamma}(\mathbf{r}) = -\frac{\partial}{\partial r_j}\{\bar{u}_j(\mathbf{r})\bar{\Gamma}(\mathbf{r}) + 1!K_j^{(1)}(\mathbf{r};R')\,G^{(1)}(\mathbf{r};R^1) + 2!K_j^{(2)}(\mathbf{r};R',R'')\,G^{(2)}(\mathbf{r};R',R'')\}, \tag{6.4}$$

$$\mathscr{L}G^{(1)}(\mathbf{r};R_1) = -\frac{\partial}{\partial r_j}\{K_j^{(1)}(\mathbf{r};R_1)\bar{\Gamma}(\mathbf{r}) + \bar{u}_j(\mathbf{r})\,G^{(1)}(\mathbf{r};R_1) + 2K_j^{(1)}(\mathbf{r};R')\,G^{(2)}(\mathbf{r};R_1,R') + 2K_j^{(2)}(\mathbf{r};R_1,R')\,G^{(1)}(\mathbf{r};R')\}, \tag{6.5}$$

$$\mathscr{L}G^{(2)}(\mathbf{r};R_1,R_2) = -\frac{\partial}{\partial r_j}\left\{K_j^{(2)}(\mathbf{r};R_1,R_2)\bar{\Gamma}(\mathbf{r}) + \bar{u}_j(\mathbf{r})\,G^{(2)}(\mathbf{r};R_1,R_2) + \frac{1}{2!}[K_j^{(1)}(\mathbf{r};R_1)\,G^{(1)}(\mathbf{r};R_2) + K_j^{(1)}(\mathbf{r};R_2)\,G^{(1)}(\mathbf{r};R_1)]\right\}. \tag{6.6}$$

Equations (6.4)–(6.6) are to be solved for $\bar{\Gamma}$ and $G^{(i)}$ with $\bar{\mathbf{u}}$ and $K^{(i)}$ given. The correlation of the fluctuation in Γ is found following methods already given.

Let us specialize as in paper III to statistically homogeneous and isotropic turbulence involving incompressible fluid flow. Let $\bar{\mathbf{u}}(\mathbf{r}) = \bar{\Gamma}(\mathbf{r}) = 0$ and all kernels be functions of the difference variables as in (2.11), letting (as there) $U^{(n)}$ [and $\gamma^{(n)}$] represent the homogeneous velocity field kernels [and reactant kernels]. Then for (3.2) let

$$\left.\begin{aligned}\Gamma^{(1)}(\mathbf{r}) &= \int \gamma_{\alpha_1}^{(1)}(\mathbf{r}-\mathbf{r}_1)\,H_{\alpha_1}^{(1)}(\mathbf{r}_1)\,d\mathbf{r}_1,\\ \Gamma^{(2)}(\mathbf{r}) &= \iint \gamma_{\alpha_1\alpha_2}^{(2)}(\mathbf{r}-\mathbf{r}_1,\mathbf{r}-\mathbf{r}_2)\,H_{\alpha_1\alpha_2}^{(2)}(\mathbf{r}_1,\mathbf{r}_2)\,d\mathbf{r}_1 d\mathbf{r}_2,\\ &\vdots\end{aligned}\right\} \tag{6.7}$$

where $\gamma^{(2)}$ is symmetric in the subscripts 1 and 2. Equations (6.5) and (6.6) become

$$\mathscr{L}\gamma_\alpha^{(1)}(\mathbf{r}) = -\frac{\partial}{\partial r_j}\left\{2\int U_{j\beta}^{(1)}(\mathbf{r}')\,\gamma_{\alpha\beta}^{(2)}(\mathbf{r},\mathbf{r}')\,d\mathbf{r}' + 2\int U_{j\alpha\beta}^{(2)}(\mathbf{r},\mathbf{r}')\,\gamma_\beta^{(1)}(\mathbf{r}')\,d\mathbf{r}'\right\}, \tag{6.8}$$

$$\mathscr{L}\gamma_{\alpha_1\alpha_2}^{(2)}(\mathbf{r}_1,\mathbf{r}_2) = -\left(\frac{\partial}{\partial r_{1j}} + \frac{\partial}{\partial r_{2j}}\right)\frac{1}{2!}[U_{j\alpha_1}^{(1)}(\mathbf{r}_1)\,\gamma_{\alpha_2}^{(1)}(\mathbf{r}_2) + U_{j\alpha_2}^{(1)}(\mathbf{r}_2)\,\gamma_{\alpha_1}^{(1)}(\mathbf{r}_1)], \tag{6.9}$$

with in $\mathscr{L}$ of (6.9)

$$\nabla^2 = \left(\frac{\partial}{\partial r_{1\alpha}} + \frac{\partial}{\partial r_{2\alpha}}\right)\left(\frac{\partial}{\partial r_{1\alpha}} + \frac{\partial}{\partial r_{2\alpha}}\right).$$

Equations (6.8) and (6.9) are the appropriate ones for the solution of mixing problems to lowest order of non-Gaussianity when the process is statistically homogeneous and isotropic. Isotropy is used to simplify the tensors in the usual way. The correlation of Γ is to this order,

$$\langle\Gamma(\mathbf{r}')\Gamma(\mathbf{r}'')\rangle = \langle\Gamma^{(1)}(\mathbf{r}')\Gamma^{(1)}(\mathbf{r}'')\rangle + \langle\Gamma^{(2)}(\mathbf{r}')\Gamma^{(2)}(\mathbf{r}'')\rangle, \tag{6.10}$$

† The contribution of the product of the (Gaussian) fluctuation terms to the mean value of the concentration was dropped in equation (2.2) of paper III. This caused no difficulty there for the later work on homogeneous turbulence.

where, using (6.7), we find with $\mathbf{r} \equiv \mathbf{r}'' - \mathbf{r}'$

$$\langle \Gamma^{(1)}(\mathbf{r}')\Gamma^{(1)}(\mathbf{r}'')\rangle = \int \gamma_\alpha^{(1)}(\mathbf{r}_1)\,\gamma_\alpha^{(1)}(\mathbf{r}+\mathbf{r}_1)\,d\mathbf{r}_1, \tag{6.11}$$

$$\langle \Gamma^{(2)}(\mathbf{r}')\Gamma^{(2)}(\mathbf{r}'')\rangle = 2\iint \gamma^{(2)}_{\alpha_1\alpha_2}(\mathbf{r}_1,\mathbf{r}_2)\,\gamma^{(2)}_{\alpha_1\alpha_2}(\mathbf{r}+\mathbf{r}_1,\mathbf{r}+\mathbf{r}_2)\,d\mathbf{r}_1 d\mathbf{r}_2. \tag{6.12}$$

If we define, after paper III,

$$G(k^2) = (2\pi)^{-3}\int e^{-i\mathbf{k}\cdot\mathbf{r}}\langle \Gamma(\mathbf{r}')\Gamma(\mathbf{r}'')\rangle\, d\mathbf{r}, \tag{6.13}$$

and then let G_1 be the transform of (6.11) [and G_2 of (6.12)], we have the simple relation,

$$G_1(k^2) = (2\pi)^3\,\gamma_\alpha^{(1)}(\mathbf{k})\,\gamma_\alpha^{(1)*}(\mathbf{k}) \tag{6.14}$$

with

$$\gamma^{(1)}(\mathbf{k}) \equiv (2\pi)^{-3}\int e^{-i\mathbf{k}\cdot\mathbf{r}}\gamma_\alpha^{(1)}(\mathbf{r})\,d\mathbf{r}. \tag{6.15}$$

As before the Gaussian kernel is the square root of the energy spectrum function. The G_2 is similarly obtained from (6.12) and (6.13).

In connexion with the moments there is a remark which can be made. In paper III it is noted that the non-linear term in (6.1) conserves concentration 'energy' $\langle\Gamma^2\rangle$. For a statistically homogeneous, incompressible process there is a more general conservation result. Consider time changes caused by the inertial term

$$\frac{\partial}{\partial t_I}\Gamma \equiv -u_j\frac{\partial}{\partial x_j}\Gamma. \tag{6.16}$$

Multiply this by $dF(\Gamma)/d\Gamma$, where F is any sufficiently well-behaved function and average. We find from (6.16)

$$\begin{aligned}\frac{\partial}{\partial t_I}\langle F(\Gamma)\rangle &= -\left\langle\left(\frac{d}{d\Gamma}F\right)u_j\frac{\partial}{\partial r_j}\Gamma\right\rangle \\ &= -\left\langle u_j\frac{\partial}{\partial r_j}F(\Gamma)\right\rangle \\ &= 0.\end{aligned} \tag{6.17}$$

For instance, the inertial term preserves all moments ($F = \Gamma^n$) in homogeneous flow.†

We discuss the relation of the expansion leading to (6.8) and (6.9) to the quasi-normal approximation given in paper III. The remarks of § 5 of this paper apply with little modification. We shall not reproduce here the complete set of equations of paper III; the reader can refer to that paper for details. The authors of III find, proceeding with the usual quasi-normal approximation,

$$\{(\partial/\partial t) + [2Dk^2 + 2C]\}G(k^2,t) = T(k^2,t) \tag{6.18}$$

and the transfer is given by an integral over L determined from

$$\{(\partial/\partial t) + f\}L(k,k'',\mu',t) = 2k^2(1-\mu^2)\,\phi_1(k'^2)[G_1(k''^2,t) - G_1(k^2,t)] \tag{6.19}$$

† The Burgers' model of paper I also has this property, for its inertial term permits a proof like (6.17). Indeed the inertial term of the Navier–Stokes equation for the same reason not only conserves energy, u^2, but conserves any function of u^2. All of these remarks assume statistical homogeneity.

with ϕ_1 defined implicitly by,

$$\phi_{jk}(\mathbf{k}) = \phi_1(k^2)P_{jk}(\mathbf{k}) \tag{6.20}$$

and ϕ_{jk} the Fourier transform of the (given) second-order velocity correlation. The right-hand side of (6.19) is obtained through the use of the quasi-normal approximation. It has the characteristic quadratic form of this method. We have added the subscript 1 to the spectral quantities in the transfer.

Consider now the result of the treatment of the present paper when applied to the moment equations of paper III. Equation (6.18) is unchanged, except that G is the sum of the Gaussian term G_1 [see (6.14)] and the lowest-order non-Gaussian term G_2. The transfer term so obtained is the same as that from (6.19), taken from paper III. It is emphasized that the functions in that term are however the Gaussian ones. The difference between these two formulations, for calculating the spectrum functions, is slight in many situations. For the validity of either method we require that the transfer term remain small in dimensionless form. Otherwise it would be necessary to take into account at least one higher-order term in the Wiener expansion if the calculation is to be valid at every step. The negative energy spectra obtained by O'Brien & Francis occur when this basic criterion is violated. To see this, estimate the transfer term, the time rate of change of G, from their figure 1. There it is found that initially the transfer term is small and the time rate of change correspondingly small. But at later times, see changes in going from $\tau = 1{\cdot}02$ to $\tau = 1{\cdot}98$, the time rate of change of G is of order unity. Hence the non-Gaussian term is of the order of the Gaussian one and the simple use of the lowest-order terms in Wiener expansion, approximated by the quasi-normal hypothesis, is no longer valid. In fact at slightly later values of time the quasi-normal spectrum function becomes negative, at a wave-number of about $1{\cdot}3$. For an energy spectrum function more nearly approximating the physical one initially [the authors of paper III used $\phi(k^2, 0) = (3\pi^{\frac{1}{2}})^{-1}k^2e^{-k^2}$] the calculation might be reliable to much later times. But here again the expansion of the present paper is more complete and probably simpler and, as suggested at the end of § 5, more stable.

7. Comparison with experiment

In this section we describe calculations for homogeneous and isotropic turbulence. The calculations are based on the truncated equations (3.13S) and (3.14S). These equations constitute an initial value problem. The initial values are set by wind tunnel experiments. Calculated values are then compared with flow measurements farther downstream.

Before discussing the calculation and experiments, consider a hallmark of the truncated representation. All of the moments of the velocity field can be obtained, to lowest order, from the two lowest-order kernels. For instance, moments of the field at any number of points can be so expressed. One way to state the restriction of the truncation is the following. Any one-argument characteristic of the fields, e.g. the energy spectrum function, together with any other characteristic which is a function of two arguments, e.g. the triple-velocity correlation function taken at three points, is sufficient to determine all other characteristics

of the velocity and pressure fields. Thus truncating the random expansion after the second term amounts to asserting that all characteristics of the fields can be approximately expressed as functionals of the two low-order velocity correlations. Analogous statements can be made for higher-order truncations.

Consider an incompressible fluid which initially is in a state of statistically homogeneous and isotropic flow. To be precise we should define an ensemble of such flows with these properties. The flow is then allowed to decay, force-free. We wish to find the statistical characteristics at later time. As usual, we use wind tunnel experiments to approximate this ideal flow problem. As fluid passes the grid in the tunnel, it comes to a statistical state which corresponds to the initial state of the idealized flow. Then, as the fluid moves downstream, it decays somewhat like the fluid in the idealized experiment. It is customary to treat the distance downstream as proportional to the time following the initial disturbance.

We need kernel initial values (our initial time is $t = 1{\cdot}5$ or $x/M = 30$),

$$U^{(1)}_{ij}(\mathbf{r}; t = 1{\cdot}5) \quad \text{and} \quad U^{(2)}_{ijm}(\mathbf{r}_1, \mathbf{r}_2; t = 1{\cdot}5). \tag{7.1}$$

Hence, at $t = 1{\cdot}5$ we need two characteristics of the flow, one a function of one variable and the other a function of two variables. For the first we use the energy spectrum function measured by Stewart & Townsend (1951) at $x/M = 30$. The second is harder to find. It could be the triple correlation at three points. That characteristic would be very interesting for this theory but is not available. We content ourselves with the triple correlation at two points measured in a similar experiment by Stewart (1951). Then $U^{(2)}$, a function of two space variables, is not fully determined initially by this measurement (a function of but one variable). We need a restricting hypothesis and use one of a type from which good results for Burgers' model equation were obtained in paper I. We suppose that initially

$$U^{(2)}_{i\alpha_1\alpha_2}(\mathbf{k}_1, \mathbf{k}_2; t = 1{\cdot}5) = \tfrac{1}{2} i a_0 P_{i\gamma\delta}(\mathbf{k}_1 + \mathbf{k}_2) U^{(1)}_{\gamma\alpha_1}(\mathbf{k}_1; 1{\cdot}5) U^{(1)}_{\delta\alpha_2}(\mathbf{k}_2; 1{\cdot}5). \tag{7.2}$$

The constant a_0 is found by adjusting the theoretical initial triple correlation of (4.11) and (4.14), using (7.2) for the initial value of $U^{(2)}$. The constant a_0 is adjusted to fit Stewart's experimental values at $x/M = 30$.

One might at first wonder if this attempt to fit the initial state of the turbulence could be successful. We have the function $U_1(k; t = 1{\cdot}5)$ and the single parameter a_0 to be adjusted by trial and error to fit the spectrum function $E(k, 1{\cdot}5)$ *and* the (normalized) triple correlation $h(r)$ at $t = 1{\cdot}5$. The initial fit is fairly good, as can be seen in figures 4 and 5. In view of the experimental and theoretical uncertainties it was decided not to carry the initial fitting process beyond the stage shown. We use $a_0 = 5^{-\frac{1}{2}}$ here. The dissipation length λ is given by

$$\lambda^2 = [-Q''(0)/Q(0)]^{\frac{1}{2}}, \tag{7.3}$$

where $Q(r)$ is the longitudinal velocity-correlation (note $Q(0) = u^2$). In effect a single parameter a_0 has fitted the initial value of a whole function, $h(r, t = 1{\cdot}5)$. This seems to indicate that the actual value of the non-Gaussian part, $U^{(2)}$, must be approximately given by (7.2) at the initial time. A measurement of the triple correlation at three points would be very interesting as a check on this hypothesis. One reason for the success of (7.2) may be this: we see in (3.14*S*)

that, if a process is initially purely Gaussian, then for moderate times (say to $t = 0{\cdot}5$) the non-Gaussian part is approximately the product of the time and the right-hand side, i.e. the same as the hypothesis (7.2). In figure 4 we see that initially

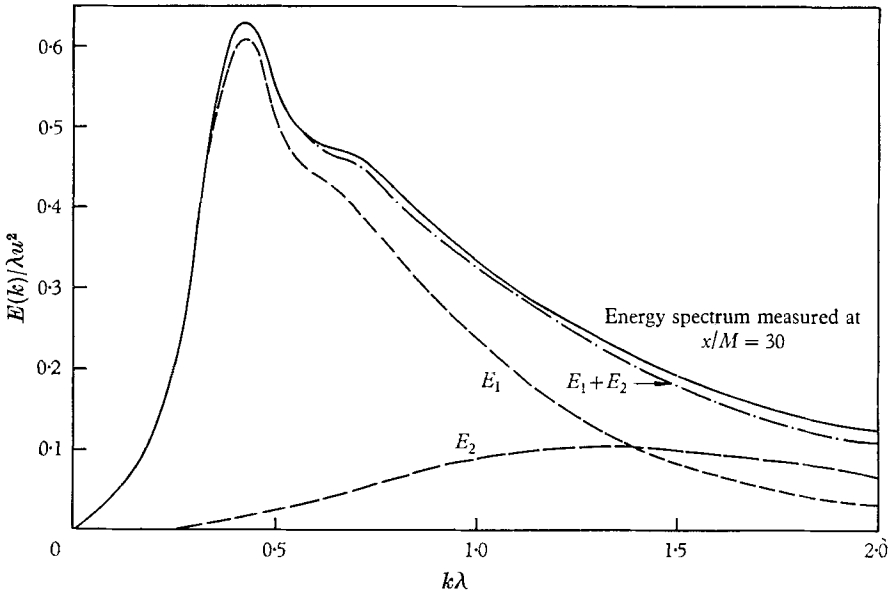

FIGURE 4. Initial E_1, Gaussian part and E_2, non-Gaussian part. $R_M = 5300$ (data from Stewart & Townsend 1951).

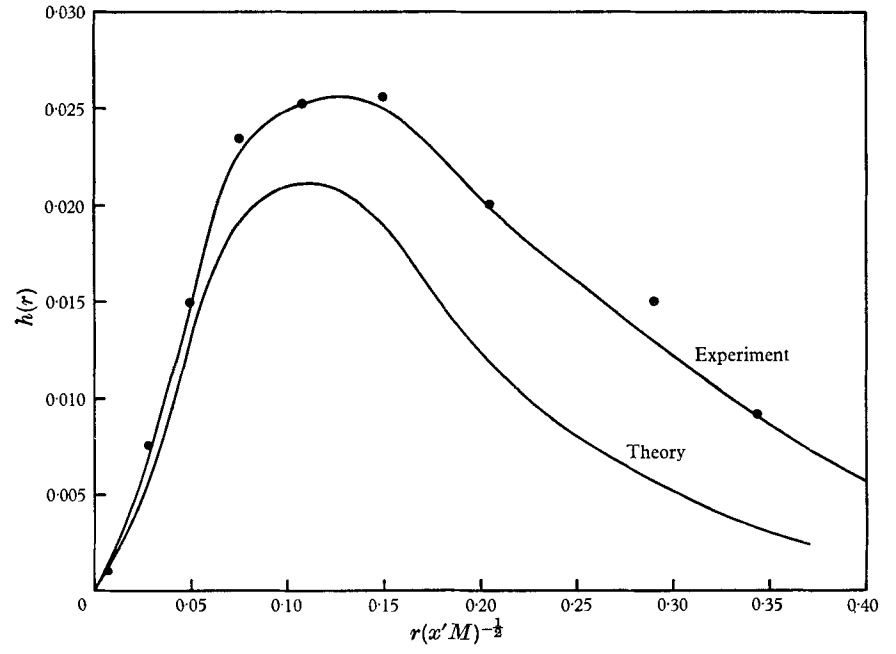

FIGURE 5. Initial triple correlation $h(r) = -\langle u^2u'^2\rangle/2u^3$, $x' = x+5M$, $R_M = 5300$ (data from Stewart 1951).

the total lowest-order, non-Gaussian energy ($\int E_2 dk$) is 10–20 % of the Gaussian part, consistent with the assumption of nearly normal flow. The two spectra become the same size for wave-numbers equal to a few times the energy-spectrum-maximum wave-number. The assumption is that, though this is so for these larger wave-numbers, the next highest order, E_3 (formed from the process cubic in the white noise function) is negligible compared with the two lower-order ones. The expansion is such that it is in principle possible to check this assumption by direct computation [see (3.15)].

We now construct the equations used in the numerical integrations for the determination of the turbulence characteristics at later time (after $t = 1{\cdot}5$). Integrate (3.14S) in time [use (7.2) for the initial value of $U^{(2)}$] and substitute the result for the second kernel in (3.13S). The angular integrals can be performed and we find for U_1 the equation

$$\left(\frac{\partial}{\partial t}+\nu k_1^2\right) U_1(k_1,t) = -(2\pi)^{-3} k_1^5 \int_0^\infty dx \left\{\int_{t_0}^t dt' U_1(k_1,t') \times U_1(k_1 x,t') \exp\{-\nu k_1^2(1+x^2)(t-t')\} F(x,\nu k_1^2(t-t')) + a_0 U_1(k_1,t_0) U_1(k_1 x,t_0) \exp\{-\nu k_1^2(1+x^2)(t-t_0)\} F(x,\nu k_1^2(t-t_0))\right\}, \quad (7.4)$$

with
$$F(x,b) \equiv \int_{-1}^1 dy \frac{x^2(1-y^2)(1+2xy-x^3y)}{1+x^2+2xy} e^{-2xyb}$$
$$= e^{b(1+x^2)} \frac{x(1-x^2)^3}{16} \int_{-b(1-x)^2}^{-b(1+x)^2} \frac{e^u}{u} du + [(x^5/8b)-(x/8b)+(x/8b^2)-(x/4b^3)-(1/2b^3x)] \sinh 2bx + [(x^4/4b)-(x^2/4b)+(1/b^2)-(x^2/2b^2)] \cosh 2bx. \quad (7.5)$$

The Gaussian part of the energy spectrum, E_1, is obtained by substituting U_1 found from (7.3) in (4.5). Once U_1 is found, the kernel $U^{(2)}$ is obtained by integrating (3.14S). The non-Gaussian part of the spectrum, E_2, is found by substitution in (4.6). The computations were performed on the University of Minnesota Control Data 1604 digital computer. It should be emphasized that the fields are calculated from their initial values without the use of adjustable parameters. The initial fluctuation Reynolds number used in the calculations was 250 (about the same as that of the experiments). There is an important energy check on the correctness of the calculations. We first obtain $E(k)$ using (4.4)–(4.6) by multiplying (3.13S) by $U^{(1)}$ and adding to it the integral of the product of (3.14S) and $U^{(2)}$. We integrate E to construct the total energy $\int_0^\infty E(k) dk$. It is of course found that the transfer term drops out (conservation of energy). The Taylor energy relation follows (see Batchelor 1953),

$$\frac{\partial}{\partial t}\int_0^\infty [E_1(k)+E_2(k)] dk = -2\nu \int_0^\infty k^2 (E_1+E_2) dk. \quad (7.6)$$

This relation can be used as a check on the numerical results. For the work reported here (7.6) was fulfilled to within 10 % initially; at $t = 6$ it was fulfilled to within about 30 %. Evidently (7.6) depends strongly on the large wave-number

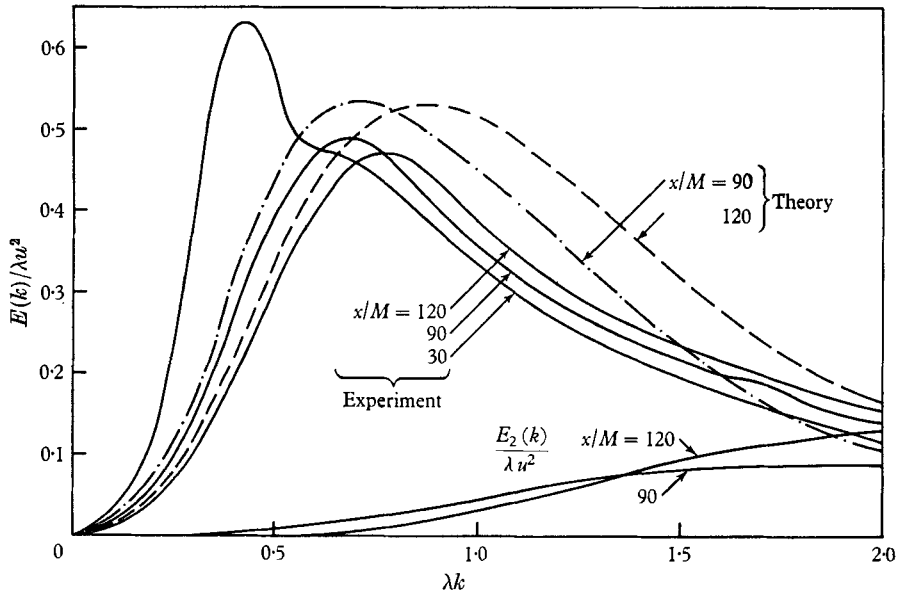

FIGURE 6. Energy spectrum downsteam (data from Stewart & Townsend 1951).

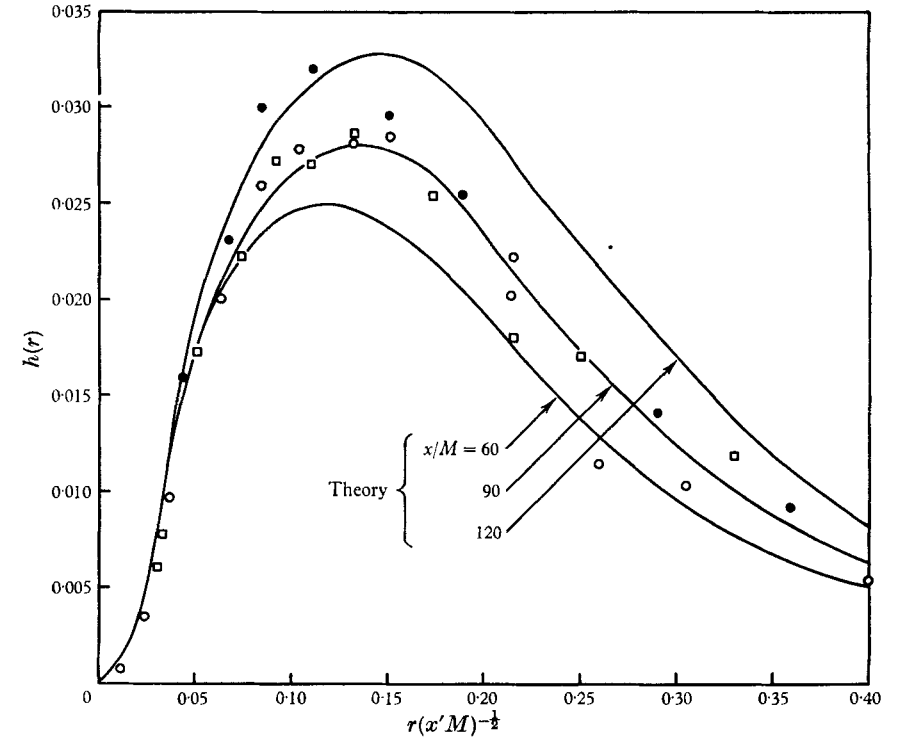

FIGURE 7. Triple velocity correlation. ○, $x/M = 60$; □, $x/M = 90$; ●, $x/M = 120$; $x' = x + 5M$ (from Stewart 1951).

behaviour of the spectra, in the dissipation range. The results could be improved by taking smaller time steps. We used varying time steps. A small step, 0·005, was needed initially. Later in the calculation this was increased progressively to 0·05. It was possible to make the step in wave-number space larger: $\Delta k = 0{\cdot}1$ was used.

Calculated results at distances of $x/M = 60$, 90 and 120 [$t = 3$, 4·5 and 6] are shown in figures 6 and 7. The calculation is of course extended far beyond a perturbation range in time (to $t = 6$). The results for the energy spectrum shown in figure 6 are fairly good, though not quantitatively perfect. (Later values of the non-Gaussian part of the spectrum are also shown in that figure.) The peak of the

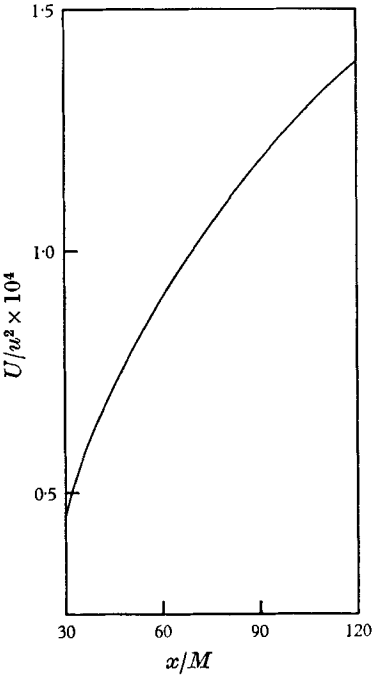

FIGURE 8. Initial-period energy decay.

spectrum has moved correctly to somewhat larger wave-numbers at later times. Of course at $x/M = 30$, the experimental and initial theoretical curves were chosen to correspond exactly. Probably the largest significant discrepancy is that of the differing slopes at the largest wave-numbers shown. This may indicate that the initial value of the non-Gaussian part of the process, measured by a_0, should be larger (as may be seen in figure 5).

The normalized triple-correlation $h(r)$ is shown for various distances downstream in figure 7. The transfer function and skewness factor can be obtained from these curves if desired. However, the triple correlation shows the general trend of these characteristics. The results shown in figure 7 are at least qualitatively encouraging. The maximum of the triple correlation increases at later times (greater distances downstream) in the experiments. The computation shows this characteristic as well. This increase is equivalently an increase in the

non-Gaussianity of the flow process. The growth of the triple correlation, that is, of the non-Gaussian part of the flow, deserves further comment. In the usual turbulence experiment and in many applications, this characteristic does not become large. However, in idealized calculations it is possible to continue for large enough times so that $\int E_2 dk$ approaches $\int E_1 dk$. In such a situation, assuming the process does not damp out first, there is little justification for truncating the Wiener expansion after two terms. Orzsag & Bissonnette (1967) have demonstrated such a failure of the nearly-normal truncated expansion in situations ultimately requiring higher-order expansions. The fit in figure 7 is not perfect, but fairly good, making allowances for possible, systematic errors in these early experiments as well as for departures from the idealizing assumptions (statistical homogeneity and isotropy).

We show the calculated initial energy decay in figure 8. The expression plotted in that figure has been found to be approximately linear in some experiments (see Batchelor 1953).

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